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AN EXTENSION OF GENERALIZED UPPER BOUNDED TECHNIQUES FOR LINEAR PROGRAMMING

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ABSTRACT

The paper 1 by Dantzig and Wolfe suggested the need for developing new techniques for solving linear programming problems with a special matrix structure. A number of techniques have appeared since then. In this report, an algorithm for solving a structured linear programming problem with a very large number of blocks is given. The main feature of the method as described in 3 is to carry out the computation with the help of a smaller basis whose order is equal to the number of linking equations coupling together the various blocks.

An Extension of Generalized Upper Bounded Techniques
for Linear Programming

1. Introduction+

The decomposition principle of Dantzig and Wolfe [1] for solving a system with a block diagonal structure is well known. An efficient computational procedure has been given by Dantzig and Van Slyke [2] [3] for solving a very large block diagonal structure where each of the blocks contains just one equation. Allowing for the possibility of several equations in each block, it would be interesting to investigate the corresponding technique for a large number of blocks. The purpose of this paper is to describe this technique. The main feature of this method as described in [2] [3] is to carry out the computation with the help of a smaller basis whose order is equal to the number of linking equations, coupling together the various blocks.

We shall concern ourselves with the problem of solving the following structured system

$$\begin{array}{llllll} A_0X_0 & + A_1X_1 + & \cdots & + A_LX_L & = b^0 \\ & B_1X_1 & & & = b^1 \\ & & \ddots & & \vdots \\ & & & B_LX_L & = b^L \end{array}$$

$$X_i \geq 0$$

+ The author is thankful to Professor R. M. Van Slyke and Mr. Paul Rech for their comments.

where the first component of the vector X_0 , unrestricted in sign, is to be maximized.

The system in the expanded form is given in Figure I.

A problem which may have this structure is a multiple plant model where each plant is represented by a different block, the blocks being coupled together by raw material allocation and product distribution [4].

We start with some definitions and theorems in the next section, and in the following the algorithm will be described. We conclude with a numerical example illustrating the application of the algorithm.

Max x_o : subject to

$$A_1^o x_o + A_1^1 x_1 + \dots + A_1^{n_o} x_{n_o} + A_1^{n_o+1} x_{n_o+1} + \dots + A_1^{n_1} x_{n_1} + A_1^{n_1+1} x_{n_1+1} + \dots + A_2^{n_2} x_{n_2} + \dots + A_1^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_1^N x_N = b_1$$

$$A_2^o x_o + A_2^1 x_1 + \dots + A_2^{n_o} x_{n_o} + A_2^{n_o+1} x_{n_o+1} + \dots + A_2^{n_1} x_{n_1} + A_2^{n_1+1} x_{n_1+1} + \dots + A_2^{n_2} x_{n_2} + \dots + A_2^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_2^N x_N = b_2$$

$$A_M^o x_o + A_M^1 x_1 + \dots + A_M^{n_o} x_{n_o} + A_M^{n_o+1} x_{n_o+1} + \dots + A_M^{n_1} x_{n_1} + A_M^{n_1+1} x_{n_1+1} + \dots + A_M^{n_2} x_{n_2} + \dots + A_M^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_M^N x_N = b_M$$

$$A_{M+1}^{n_o+1} x_{n_o+1} + \dots + A_{M+1}^{n_1} x_{n_1} = b_{M+1}$$

$$A_{M+2}^{n_o+1} x_{n_o+1} + \dots + A_{M+2}^{n_1} x_{n_1} = b_{M+2}$$

$$- - - - - - - - - - \qquad - - - -$$

$$A_{M_1}^{n_o+1} x_{n_o+1} + \dots + A_{M_1}^{n_1} x_{n_1} = b_{M_1}$$

$$A_{M_1+1}^{n_1+1} x_{n_1+1} + \dots + A_{M_1+1}^{n_2} x_2 = b_{M_1+1}$$

$$A_{M_1+2}^{n_1+1} x_{n_1+1} + \dots + A_{M_1+1}^{n_2} x_2 = b_{M_1+2}$$

$$- - - - - - - - - - - \qquad - - -$$

$$A_{M_2}^{n_1+1} x_{n_1+1} + \dots + A_{M_2}^{n_2} x_{n_2} = b_{M_2}$$

$$A_{M_{L-1}+1}^{n_{L-1}+1} x_{n_{1-1}+1} + \dots + A_{M_{L-1}+1}^N x_N = b_{M_{L-1}+1}$$

$$A_{M_{L-1}+2}^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_{M_{L-1}+2}^N x_N = b_{M_{L-1}+2}$$

$$A_{M_L}^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_{M_L}^N x_N = b_{M_L}$$

$$x_i \geq 0 \quad (i \neq 0)$$

Figure I

2. Definitions and Notations

We shall refer to the system of equations in (1) which couple together the different blocks as the "linking set". The k-th set of columns or variables S_k is a set of columns or variables that a linking set has in common with the k-th block. The set of columns S_0 does not have any block down below and therefore has all the zero entries in $(M+1)^{st}$ through M_L rows.

We assume that the whole system (1) and the different blocks are all of full rank. The basis for the system (1) is of the form

$$[A^{j_1}, \underline{A}^{j_2}, \ldots, \underline{A}^{j_{M_L}}]$$

where underscoring is used in designating the components for the full system so as to distinguish them from the first M components which are represented without underscoring. For convenience of notation, we assume that the number of equations in block k is m_k $(k = 1, \ldots, L)$.

Theorem 1: *Every basis of (1) must have at least m_p columns from the set S_p.*

Proof. Let

$$B_f = (\underline{A}^{j_1}, \underline{A}^{j_2}, \ldots, \underline{A}^{j_{M_L}})$$

be the basis for the full system. Now consider any other M_L- vector

$$(0, 0, \ldots, 0, b_{M_p+1}, b_{M_p+2}, \ldots, b_{M_{p+1}}, 0, \ldots, 0, \ldots, 0)^T , \quad 0 \le p \le L-1$$

there exist λ_j such that

$$\sum_{i=1}^{M_L} \lambda_i \underline{A}^{j_i} = (0,\ldots,0,b_{M_p+1},b_{M_p+2},\ldots,b_{M_{p+1}},0,\ldots,0)^T$$

i.e.,

$$\sum_i \lambda_i A_k^{j_i} = b_k \qquad k = M_p+1,\ M_p+2,\ldots,M_{p+1}\ ,$$

$$0 \le p \le L-1$$

which shows that there are m_{p+1} or more A^{j_i} in the (p+1)th block since each of the blocks is assumed to be of the full rank. This proves the theorem.

Theorem 2: *The number of sets $S_i (i \neq 0)$ having basic variables more than the number of equations m_i in the corresponding block cannot exceed $M-1$.*

Proof. The system (1) is of full rank; therefore, the basis consists of $M + \sum_{i=1}^{L} m_i$ vectors. But it has been shown in Theorem 1 that each set S_i contains at least m_i vectors, eliminating all the basic vectors except for M of them. One of these is the variable x_0 to be optimized; then at most, $M-1$ basic variables remain to make up sets S_i $(i \neq 0)$ with more than m_i basic variables.

Borrowing the terminology from [2], we call the set S_0 and all those sets S_i having basic variables more than the number of equations in the corresponding block as *essential* sets. All other sets are called *inessential* sets.

3. Discussion of the Algorithm

Let us suppose that we have at hand an initial basic feasible solution to

(1) which can always be obtained by using the Phase I procedure of the simplex method. This gives a division of the sets S_i $(i \neq 0)$ into essential and inessential groups as explained above. All the subproblems $B_i X^i = b^i$ that belong to the inessential sets are solved independently by setting to zero the variables not associated with the basis. Since the essential set S_t contains more basic variables than the number of equations m_t in the block associated with that set, we may regard the independent m_t columns of the basis of (1) (the existence being implied by Theorem 1) in such a set S_t as key columns and the corresponding variables as key variables. The notation A^{k_1} and x_{k_1} will be used for key column and key variable respectively. For the sake of uniformity, let us call all the columns of the basis associated with inessential sets as key columns. Having solved the subproblems corresponding to the inessential sets for key variables as stated above, we next solve the subproblems in the essential sets for the key variables by setting the remaining variables in that set to zero.

Let

$$
\underline{\mathbf{H}}^{(t)} = \begin{bmatrix}
A_1^{k_1} & A_1^{k_2} & A_1^{k_{m_t}} \\
A_2^{k_1} & A_2^{k_2} & A_2^{k_{m_t}} \\
\hdashline
A_M^{k_1} & A_M^{k_2} & A_M^{k_{m_t}} \\
 & \bigcirc & \\
A_{M_{t-1}+1}^{k_1} & A_{M_{t-1}+1}^{k_2} & A_{M_{t-1}+1}^{k_{m_t}} \\
A_{M_{t-1}+2}^{k_1} & A_{M_{t-1}+2}^{k_2} & A_{M_{t-1}+2}^{k_{m_t}} \\
\hdashline
A_{M_t}^{k_1} & A_{M_t}^{k_2} & A_{M_t}^{k_{m_t}} \\
 & \bigcirc &
\end{bmatrix} = \begin{bmatrix} \mathbf{U}^{(t)} \\ 0 \\ \mathbf{V}^{(t)} \\ 0 \end{bmatrix}
$$

be the coefficient matrix of the key variables in the set S_t . We note that the matrices $\mathbf{U}^{(t)}, \mathbf{V}^{(t)}$ correspond to the key variables in the "linking set" and the t-th block respectively.

For every non-key column $\underline{A}^j \varepsilon S_t$

$$
\underline{A}^j = (A_1^j, \ldots, A_M^j, 0, \ldots, 0, A_{M_{t-1}+1}^j, A_{M_{t-1}+2}^j, \ldots, A_{M_t}^j, 0, \ldots, 0)^T, \quad t = 1,2,\ldots,L \quad .
$$

We now write

$$
\underline{P}^j = \underline{A}^j - \mathbf{H}^{(t)} \boldsymbol{\lambda}_j^{(t)} \quad \text{---------------} \quad (2)
$$

where

$$\lambda_j^{(t)} = \left(\lambda_{1/j}^{(t)}, \lambda_{2/j}^{(t)}, \ldots, \lambda_{m_{t/j}}^{(t)}\right)^T = \mathcal{V}^{(t)-1}(A^j_{M_{t-1}+1}, A^j_{M_{t-1}+2}, \ldots, A^j_{M_t})^T \quad . \quad -- (3)$$

The superscript within parentheses is used to indicate the number of the block. Since the $(M+1)^{st}$ *through* M_L components of $\underline{P}^j$ are zero, we may also write

$$P^j = A^j - \mathcal{V}^{(t)} \lambda_j^{(t)} \quad -------- (4)$$

Repeating the above procedure for every non-key columns and transferring the key columns to the right, we obtain the reduced system

$$\sum Y_j P^j = Q \quad ---------- (5)$$

where

$$Q = b - \sum_r \mathcal{V}^{(r)} X_{k(r)} \quad . \quad ------- ---- (6)$$

Here, $X_{k(r)}$, the vector of key variables in r-th block, is determined by solving the subproblem in that block on setting non-key variables to zero.

We shall now show that the basis B for the reduced system (5) consists of the column associated with the variable x_0 and all the non-key columns left in B_f after these are modified as in (2).

Assume, therefore, that after performing the necessary operations and transfer as indicated above, we are left with the columns $p^{\ell_1}, p^{\ell_2}, \ldots, p^{\ell_M}$. We assert that these form the basis B for the system (5). If not, there exist α_k not all zero such that

$$\sum_{k=1}^{M} \alpha_k P^{\ell_k} = 0$$

i.e.,

$$\sum_{k=1}^{M} \alpha_k \underline{P}^{\ell_k} = 0$$

which, on using (2) yields

$$\sum_{i=1}^{M_L} \beta_i \underline{A}^{j_i} = 0$$

where not all β_i are zero. But this implies that $(\underline{A}^{j_1}, \underline{A}^{j_2}, \ldots, \underline{A}^{j_{M_L}})$ are dependent and hence a contradiction.

In order to generate a better solution or test optimality, we compute the price vector

$$(\Pi; \mu^{(1)}; \mu^{(2)}; \ldots; \mu^{(L)}) = (\Pi_1, \Pi_2, \ldots, \Pi_M; \mu^{(1)}_{M+1}, \mu^{(1)}_{M+2}, \ldots, \mu^{(1)}_{M_1}; \ldots; \mu^{(L)}_{M_{L-1}+1}, \ldots, \mu^{(L)}_{M_L})$$

These are determined such that

$$(\Pi; \mu^{(1)}; \mu^{(2)}; \ldots; \mu^{(L)}) \underline{A}^0 = 1 \qquad (\underline{A}^{j_1} = \underline{A}^0)$$

and

$$(\Pi; \mu^{(1)}; \mu^{(2)}; \ldots; \mu^{(L)}) \underline{A}^{j_i} = 0 \qquad i = 2, \ldots, M_L \quad .$$

Let Π^* be the first row of the inverse of the basis B for the reduced system (5), then

$$\Pi^* P^0 = 1 \qquad\qquad (P^{\ell_1} = P^0)$$

and

$$\boldsymbol{\pi}^{*} P^{\ell_k} = 0 \quad . \qquad\qquad k = 2,\ldots,M \;\text{- - - - -}\; (7)$$

Hence, $\boldsymbol{\pi}^{*}$ is a set of prices for (5). Now to compute $\boldsymbol{\mu}^{*(t)}$, $(t = 1,\ldots,L)$, we observe that for the set S_t

$$(\boldsymbol{\pi}^{*};\boldsymbol{\mu}^{*(1)};\boldsymbol{\mu}^{*(2)};\ldots;\boldsymbol{\mu}^{*(L)})\mathbf{H}^{(t)} = 0$$

i.e.,

$$\boldsymbol{\mu}^{*(t)} = -\boldsymbol{\pi}^{*}\mathbf{U}^{(t)}\mathbf{V}^{(t)-1} = -\boldsymbol{\pi}^{*}\mathbf{W}^{(t)} \qquad\qquad (t = 1,\ldots,L)$$

where

$$\mathbf{W}^{(t)} = \mathbf{U}^{(t)}\mathbf{V}^{(t)-1} \quad .$$

We now claim that $(\boldsymbol{\pi}^{*};\boldsymbol{\mu}^{*(1)};\boldsymbol{\mu}^{*(2)};\ldots;\boldsymbol{\mu}^{*(L)})$ are a set of prices for the whole system (1). For in the manner we obtained $\boldsymbol{\pi}^{*};\boldsymbol{\mu}^{*(1)};\ldots;\boldsymbol{\mu}^{*(L)}$ it is obvious that if $\underline{A}^{j_i} \varepsilon S_0$ or $\underline{A}^{j_i}$ is a key column, then

$$(\boldsymbol{\pi}^{*};\boldsymbol{\mu}^{*(1)};\boldsymbol{\mu}^{*(2)};\ldots;\boldsymbol{\mu}^{*(L)})\underline{A}^{j_1} = 1$$

$$(\boldsymbol{\pi}^{*};\boldsymbol{\mu}^{*(1)};\boldsymbol{\mu}^{*(2)};\ldots;\boldsymbol{\mu}^{*(L)})\underline{A}^{j_i} = 0 \qquad (i \neq 1) \quad .$$

On the other hand, if $\underline{A}^{j_i}$ is not a key column and $\underline{A}^{j_i} \varepsilon S_t$, then

$$(\boldsymbol{\pi}^{*};\boldsymbol{\mu}^{*(1)};\boldsymbol{\mu}^{*(2)};\ldots;\boldsymbol{\mu}^{*(L)})\underline{A}^{j_i} = \boldsymbol{\pi}^{*}A^{j_i} + \boldsymbol{\mu}^{*(t)}A^{j_i}_{(t)} \qquad \text{- - -}\; (8)$$

where $A^{j_i}_{(t)}$ is the part of the column $\underline{A}^{j_i}$ in the t-th block.

From (3), (4), (7) and (8) we obtain

$$(\pi^*;\mu^{*(1)};\mu^{*(2)};\ldots;\mu^{*(L)})\underline{A}^{j_i} = \pi^*\left(A^{j_i} - U^{(t)}\lambda^{(t)}_{j_i}\right)$$

$$= \pi^* P^{j_k} \quad \text{for some} \quad k$$

$$= 0 \; .$$

Thus $(\pi^*;\mu^{*(1)};\mu^{*(1)};\ldots;\mu^{*(L)})$ is a pricing vector for the system. Now, pricing out the non-basic columns we enter the column for which

$$\theta^{s(t)} = (\pi^*;\mu^{*(1)};\ldots;\mu^{*(L)})\underline{A}^{s(t)}$$

$$= \operatorname*{Min}_{j} \; (\pi^*;\mu^{*(1)};\ldots;\mu^{*(L)})\underline{A}^{j(t)}$$

assumes minimum for $t = 0,1,\ldots,L$ and this minimum is negative. If the minimum turns out to be non-negative, we terminate and are at the optimal solution. Assume this is not the case and the column $\underline{A}^s \epsilon S_\ell$ qualifies for entry into the basis, the next step is to find out the vector that drops out of the basis. For this purpose we express the column $\underline{A}^s$ and the vector b in terms of the current basis.

If $\overline{P^s}$ denotes the representation of P^s in terms of the basis B, then

$$P^s = B\overline{P^s} = \sum \overline{P^s_i} P^{\ell_i}$$

$$= \sum \overline{P^s_i}\,(A^{n_i} - U^{(t_i)}\lambda^{(t_i)}_{n_i})$$

where $P^{\ell_i} \epsilon S_{t_i}$ and the i-th basic variable P^{ℓ_i} of B is in column number n_i

Therefore,

$$A^s = \sum \overline{P}^s_i (A^{n_i} - \boldsymbol{U}^{(t_i)} \boldsymbol{\lambda}^{(t_i)}_{n_i}) + \boldsymbol{U}^{(\ell)} \boldsymbol{\lambda}^{(\ell)}_s \quad .$$

Hence, if

$$\underline{A}^s = \sum q^s_i \underline{A}^{j_i}$$

then

$$q^s_i = \lambda^{(\ell)}_{k_{n/s}} - \sum_{A^{n_i} \epsilon S_\ell} \lambda^{(\ell)}_{k_{n/n_i}} \overline{P}^s_i \qquad \text{if } A^{j_i} = A^{k_n(\ell)}$$

$$= - \sum_{A^{n_i} \epsilon S_t} \lambda^{(t)}_{k_{n/n_i}} \overline{P}^s_i \qquad \text{if } A^{j_i} = A^{k_n(t)}, \ t \neq \ell$$

$$= \overline{P}^s_t \qquad \text{if } A^{j_i} = A^{n_t} \text{ for some } t$$

$$= 0 \qquad \text{otherwise} \quad .$$

A similar consideration shows that

$$Q = \sum_i \overline{Q}_i P^{\ell_i}$$

$$= \sum_i \overline{Q}_i (A^{n_i} - \boldsymbol{U}^{(t_i)} \boldsymbol{\lambda}^{(t_i)}_{n_i})$$

and therefore, using (6)

$$b = \sum_i \overline{Q}_i (A^{n_i} - \boldsymbol{U}^{(t_i)} \boldsymbol{\lambda}^{(t_i)}_{n_i}) + \sum_\ell \boldsymbol{U}^{(\ell)} X_{k(\ell)}$$

This shows that if

$$\underline{b} = \sum b_i^* \underline{A}^{j_i}$$

then

$$b_i^* = x_{k_n}(\ell) - \sum_{A^{n_i} \varepsilon S_\ell} \bar{Q}_i \lambda^{(\ell)}_{k_{n/n_i}} \qquad \text{if} \quad A^{j_i} = A^{k_n(\ell)}$$

$$= \bar{Q}_t \qquad \text{if} \quad A^{j_i} = A^{n_t} \quad \text{for some } t$$

$$= 0 \qquad \text{otherwise.}$$

Let

$$\frac{b_r^*}{q_r^s} = \underset{q_i^s > 0}{\text{Min}} \frac{b_i^*}{q_i^s} \qquad i = 2, \ldots M_L$$

then the simplex method requires that $\underline{A}^{j_r}$ be dropped from the basis. We now consider the following cases.

Case 1: If $\underline{A}^{j_r}$ and $\underline{A}^s$ belong to the same set S_ℓ which is inessential, then A^s replaces A^{j_r} as key column and this does not introduce any change in the smaller basis B.

Let us denote by $\hat{Q}$ the new value of $\bar{Q}$, then

$$\hat{Q} = B^{-1}\left[b - \sum_i \mathbf{U}^{(i)} x_{k(i)} + A^{k_a(\ell)} x_{k_{a(\ell)}} - A^s x_s\right]$$

$$= \bar{Q} - B^{-1}\,(A^s x_s - A^{k_a(\ell)} x_{k_{a(\ell)}})\ , \quad A^{k_a(\ell)} = A^{j_r}\ .$$

We now determine the new set of prices and the column eligible for entry.

<u>Case 2</u> Assume $A^{j_r} \varepsilon S_m$ and is not a key column and set $P^{j_t} = A^{j_r} - \sum_i A^{k_{i(m)}} \lambda_{k_{i/j_r}}$.

The updating of B^{-1} in this case is accomplished by pivoting on the element that lies in the P^s column and in the row of P^{j_t}. We observe that

$$P = k_1 P^{\ell_1} + \ldots + k_r P^{\ell_r} + \ldots + k_M P^{\ell_M}$$

i.e.,

$$P^{\ell_r} = -\frac{k_1}{k_r} P^{\ell_1} - \ldots + \frac{1}{k_r} P^s - \ldots - \frac{k_M}{k_r} P^{\ell_M}$$

where the column A^{j_r} is the column P^{ℓ_r} in the smaller basis.

Hence, $B^{*-1} = E\, B^{-1}$ where

$$E = \begin{bmatrix} 1 & & -k_{1/k_r} & & & \\ & 1 & -k_{2/k_r} & & & \\ & & \ddots \vdots & & & \\ & & 1/k_r & & & \\ & & \vdots & \ddots & & \\ & & -k_{M/k_r} & & & 1 \end{bmatrix}$$

$\bar{Q}$ is updated by premultiplication with the matrix E.

<u>Case 3</u> If $A^{j_r} \varepsilon S_h$ and is a key column, then in the set S_h we make some column of the basis B, say, A^r, key in place of A^{j_r}. The existence of at least one such column is implied by Theorem 1. The columns of B in the set S_h have the form:

Before A^{j_r} is dropped

$$\overbrace{A^p - \lambda_{1/p} A^{k_1} - \ldots - \lambda_{j_{r/p}} A^{j_r} - \ldots - \lambda_{m(h)/p} A^{k_{m(h)}}}^{B^{\ell_p}}, \ldots, \overbrace{A^r - \lambda_{1/r} A^{k_1} - \ldots - \lambda_{j_{r/r}} A^{j_r} - \ldots - \lambda_{m(h)/r} A^{k_{m(h)}}}^{B^{\ell_r}}, \ldots, \overbrace{A^q - \lambda_{1/q} A^{k_1} - \ldots - \lambda_{j_{r/q}} A^{j_r} - \ldots - \lambda_{m(h)/q} A^{k_{m(h)}}}^{B^{\ell_q}}$$

After A^{j_r} is dropped and A^r key ($\lambda_{j_{r/r}} \neq 0$)

$$\overbrace{A^p - \left(\lambda_{1/p} - \frac{\lambda_{1/r}\lambda_{j_{r/p}}}{\lambda_{j_{r/r}}}\right) A^{k_1} - \ldots - \frac{\lambda_{j_{r/p}}}{\lambda_{j_{r/r}}} A^r - \ldots - \left(\lambda_{m(h)/p} - \frac{\lambda_{m(h)/r}\lambda_{j_{r/p}}}{\lambda_{j_{r/r}}}\right) A^{k_{m(h)}}}^{\tilde{B}^{\ell_p}}, \ldots, \overbrace{A^{j_r} - \frac{1}{\lambda_{j_{r/r}}} A^r + \frac{\lambda_{1/r}}{\lambda_{j_{r/r}}} A^{k_1} + \ldots + \frac{\lambda_{m(h)/r}}{\lambda_{j_{r/r}}} A^{k_{m(h)}}}^{\tilde{B}^{\ell_r}}, \ldots, \overbrace{A^q - \left(\lambda_{1/q} - \frac{\lambda_{1/r}\lambda_{j_{r/q}}}{\lambda_{j_{r/r}}}\right) A^{k_1} - \ldots - \frac{\lambda_{j_{r/q}}}{\lambda_{j_{r/r}}} A^r - \ldots - \left(\lambda_{m(h)/q} - \frac{\lambda_{m(h)/r}\lambda_{j_{r/q}}}{\lambda_{j_{r/r}}}\right) A^{k_{m(h)}}}^{\tilde{B}^{\ell_q}}$$

we find that $\tilde{B}^{\ell_p}, \ldots, \tilde{B}^{\ell_r}, \ldots, \tilde{B}^{\ell_q}$

$$B^{\ell_p}, \ldots, B^{\ell_r}, \ldots, B^{\ell_q} = \begin{bmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\lambda_{j_{r/p}}}{\lambda_{j_{r/r}}} & \ldots & \dfrac{-1}{\lambda_{j_{r/r}}} & \ldots & -\dfrac{\lambda_{j_{r/q}}}{\lambda_{j_{r/r}}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & 0 & & 1 \end{bmatrix}$$

Therefore, the relation between the old basis B and the new basis $\tilde{B}$ obtained in changing the key variable can be expressed as

$$\tilde{B} = BM$$

where

$$
M = \begin{bmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
0 \; 0 \cdots -\dfrac{\lambda_{j_{r/p}}}{\lambda_{j_{r/r}}} & \cdots & \cdots & \dfrac{-1}{\lambda_{j_{r/r}}} & \cdots & -\dfrac{\lambda_{j_{r/q}}}{\lambda_{j_{r/r}}} & 0 & 0 \\
 & & & & \ddots & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & 1
\end{bmatrix}
$$

It is obvious that

$$M^{-1} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ 0 & 0 \cdots\cdots & -\lambda_{j_{r/p}} & \cdots\cdots & -\lambda_{j_{r/r}} & \cdots\cdots & -\lambda_{j_{r/q}} & \cdots\cdots 0 \cdots\cdots & 0 \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & 1 \end{bmatrix}.$$

Hence, changing the key column amounts to premultiplying the old B^{-1} by M^{-1} to obtain the new $\tilde{B}^{-1}$.

Now A^{j_r} is a non-key column in the set S_h, and updating of $\tilde{B}^{-1}$ and Q are accomplished as in Case 2. Therefore, we are ready for the next iteration.

We next consider the following example illustrating the above procedure:

Example:

Max x_0

Subject to:

$$4x_0 + x_1 + 2x_2 - x_3 + x_4 - 10x_5 + 3x_6 - 4x_7 + 2x_8 + 0x_9 + 2x_{10} = 12$$

$$0x_0 + 7x_1 + x_2 + 4x_3 - 3x_4 + 18x_5 - 6x_6 + x_7 - x_8 + x_9 - x_{10} = 2$$

$$2x_0 - x_1 + 2x_2 - x_3 + x_4 - 2x_5 + 0x_6 - 3x_7 + 2x_8 - x_9 - 0x_{10} = 7$$

$$3x_2 - x_3 + x_4 - 0x_5 = 5$$

$$-4x_2 + x_3 + 8x_4 + 16x_5 = 20$$

$$0x_6 + 3x_7 - x_8 + x_9 - 0x_{10} = 1$$

$$3x_6 - 4x_7 + x_8 + 0x_9 + 0x_{10} = 2$$

$$6x_6 - 10x_7 + 3x_8 + 2x_9 - 2x_{10} = 7$$

$$x_i \geq 0 \quad (i \neq 0)$$

Assume that we have an initial basic feasible solution and the columns $\underline{A}^0$, $\underline{A}^2$, $\underline{A}^3$, $\underline{A}^4$, $\underline{A}^6$, $\underline{A}^7$, $\underline{A}^8$, $\underline{A}^9$ are in the basis.

Take $\underline{A}^2$, $\underline{A}^3$ key in the set S_1, and $\underline{A}^6$, $\underline{A}^7$, $\underline{A}^8$ key in the set S_2.

$$B = (A^0,\ A^4 - \lambda^{(1)}_{1/4} A^2 - \lambda^{(1)}_{2/4} A^3,\ A^9 - \lambda^{(2)}_{1/9} A^6 - \lambda^{(2)}_{2/9} A^7 - \lambda^{(2)}_{3/9} A^8)$$

where

$$\lambda_4^{(1)} = \begin{bmatrix}\lambda_{1/4}^{(1)}, & \lambda_{2/4}^{(1)}\end{bmatrix}^T = \begin{bmatrix}3 & -1\\ -4 & 1\end{bmatrix}^{-1} \begin{bmatrix}1\\ 8\end{bmatrix} = \begin{bmatrix}-9\\ -28\end{bmatrix}$$

and

$$\lambda_9^{(2)} = \begin{bmatrix}\lambda_{1/9}^{(2)}, & \lambda_{2/9}^{(2)}, & \lambda_{3/9}^{(2)}\end{bmatrix} = \begin{bmatrix}0 & 3 & -1\\ 3 & -4 & 1\\ 6 & -10 & 3\end{bmatrix}^{-1} \begin{bmatrix}1\\ 0\\ 2\end{bmatrix}$$

$$= \begin{bmatrix}4/3\\ 3\\ 8\end{bmatrix} .$$

Hence

$$B = \begin{bmatrix}4 & -9 & -8\\ 0 & 118 & 14\\ 2 & -9 & -8\end{bmatrix}$$

and

$$B^{-1} = \begin{bmatrix}1/2 & 0 & -1/2\\ -7/409 & 4/409 & 14/409\\ 59/409 & -9/818 & -118/409\end{bmatrix} .$$

The first row of B^{-1} gives

$$\pi^* = (1/2,\ 0,\ -1/2) .$$

Therefore,

$$\mu^{*(1)} = -(1/2,\ 0,\ -1/2) \begin{bmatrix}2 & -1\\ 1 & 4\\ 2 & -1\end{bmatrix} \begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix} = 0$$

$$\mu^{*(2)} = -(1/2,\ 0,\ -1/2) \begin{bmatrix}3 & -4 & 2\\ -6 & 1 & -1\\ 0 & -3 & 2\end{bmatrix} \begin{bmatrix}2/3 & -1/3 & 1/3\\ 1 & -2 & 1\\ 2 & -6 & 3\end{bmatrix}$$

$$= (1/2,\ -1/2,\ 0) .$$

Hence

$$(\boldsymbol{\pi}^*; \boldsymbol{\mu}^{*(1)}; \boldsymbol{\mu}^{*(2)}) = (1/2,\ 0,\ -1/2,\ 0,\ 0,\ -1/2,\ -1/2,\ 0)$$

$$\underset{j}{\mathrm{Min}}\ (\boldsymbol{\pi}^*; \boldsymbol{\mu}^{*(1)}; \boldsymbol{\mu}^{*(2)})\ \underline{A}^j = (\boldsymbol{\pi}^*; \boldsymbol{\mu}^{*(1)}; \boldsymbol{\mu}^{*(2)})\ \underline{A}^5$$

Therefore, column $\underline{A}^5$ will be introduced into the basis.

Now

$$P^5 = A^5 - \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \lambda^{(1)}_{1/5} \\ \lambda^{(1)}_{2/5} \end{bmatrix}$$

where the vector $\left[\lambda^{(1)}_{1/5},\ \lambda^{(1)}_{2/5}\right]$ as usual is chosen such that when the linear combination of the keys $\underline{A}^2$, $\underline{A}^3$ with weights $-\lambda^{(1)}_{1/5}$ and $-\lambda^{(1)}_{2/5}$ respectively is added to the column A^5, the components of A^5 in the first block vanish. Therefore,

$$\left[\lambda^{(1)}_{1/5},\ \lambda^{(1)}_{2/5}\right] = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 0, & 16 \end{bmatrix} = \begin{bmatrix} -16, & -48 \end{bmatrix}$$

and $$P^5 = A^5 - \lambda^{(1)}_{1/5} A^2 - \lambda^{(1)}_{2/5} A^3 = (-26,\ 226,\ -18)^T \quad - - - - (9)$$

Now $$\overline{P}^5 = B^{-1} (A^5 + 16A^2 + 48A^3)$$

$$= \begin{bmatrix} 1/2 & 0 & -1/2 \\ -7/409 & 4/409 & 14/409 \\ 59/409 & -9/818 & -118/409 \end{bmatrix} \begin{bmatrix} -26 \\ 226 \\ -18 \end{bmatrix} - \begin{bmatrix} -4 \\ 834/409 \\ -854/818 \end{bmatrix}$$

i.e., $$\underline{A}^5 = -4\underline{A}^0 + \frac{962}{409}\underline{A}^2 + \frac{3720}{409}\underline{A}^3 + \frac{834}{409}\underline{A}^4 + \frac{1708}{1227}\underline{A}^6 + \frac{1281}{409}\underline{A}^7 + \frac{3416}{409}\underline{A}^8 - \frac{427}{409}\underline{A}^9 \quad (10)$$

which gives representation of $\underline{A}^5$ in terms of the basis for the full system.

Also,

$$Q = [12, 2, 7]^T \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 3 & -4 & 2 \\ -6 & 1 & -1 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

where

$$[x_2, x_3] = [-25, -80] \ , \ [x_6 \ , \ x_7, \ x_8] = [7/3, 4, 11]$$

are obtained on solving the subproblems separately.

$$Q = [-31, 368, -33]^T$$

Thus

$$B^{-1}[b - \sum x_{k_\ell} A^{k_\ell}] = (1, 3, 1)^T$$

which gives

$$\underline{b} = \underline{A}^0 + 2\underline{A}^2 + 4\underline{A}^3 + 3\underline{A}^4 + \underline{A}^6 + \underline{A}^7 + 3\underline{A}^8 + \underline{A}^9 \quad .$$

hence

$$\bar{\underline{b}} = [1, 2, 4, 3, 1, 1, 3, 1]^T \quad . \qquad (11)$$

From (10) and (11) we find that $\underline{A}^7$ drops out of the basis. Since $\underline{A}^7$, the second column of S_2, is the key column, we first make some other non-key column, say A^9, as the key column. The new inverse of the coefficient matrix $V^{(2)}$ corresponding to the new key variables associated with the second block is

$$\mathbf{V}^{(2)^{-1}} = \begin{bmatrix} 2/9 & 5/9 & -1/9 \\ -2/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \end{bmatrix} \quad .$$

Also, the new inverse of the basis $\tilde{B}^{-1}$ is

$$\tilde{B}^{-1} = M^{-1}B^{-1}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \qquad \left(\text{Note: } \frac{-1}{27/9} = -1/3\right)$$

Thus,

$$\tilde{B}^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ -7/409 & 4/409 & 14/409 \\ -177/409 & 27/818 & 354/409 \end{bmatrix} \tag{12}$$

From (9) and (12) we have

$$\tilde{B}^{-1}P^5 = [-4,\ 834/409,\ 1281/409]^T \quad .$$

Pivoting on the third component, the new inverse basis

$$\hat{B}^{-1} = \begin{bmatrix} 1 & 0 & 1636/1281 \\ 0 & 1 & -834/1281 \\ 0 & 0 & 409/1281 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & -1/2 \\ -7/409 & 4/409 & 14/409 \\ -177/409 & 27/818 & 354/409 \end{bmatrix}$$

$$= \begin{bmatrix} -45/854 & 18/427 & 517/854 \\ 113/427 & -5/427 & -226/427 \\ -59/427 & 9/854 & 118/427 \end{bmatrix} \quad .$$

We again compute the new set of prices

(-45/854, 18/427, 517/854, -166/427, -2/427, 2531/2562, 1439/2562, -544/2562)

but we note that in S_2 , A^6, A^8, A^9 are key and

$$V^{(2)^{-1}} = \begin{bmatrix} 2/9 & 5/9 & -1/9 \\ -2/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \end{bmatrix} \quad .$$

Updating Q , we find

$$Q = b - \sum x_{k_\ell} A^{k_\ell}$$

$$= b - w^{(1)}(5,20)^T - w^{(2)}(1,2,7)^T$$

$$= (-61/3,\ 1048/3,\ -67/3)^T \quad ,$$

and

$$B^{-1}\ [b - \sum x_{k_\ell} A^{k_\ell}]$$

$$= \begin{bmatrix} -45/854 & 18/427 & 517/854 \\ 113/427 & -5/427 & -226/427 \\ -59/427 & 9/854 & 118/427 \end{bmatrix} \begin{bmatrix} -61/3 \\ 1048/3 \\ -67/3 \end{bmatrix} = \begin{bmatrix} 2917/1281 \\ 1003/427 \\ 409/1281 \end{bmatrix}$$

and hence

$$\underline{b} + 25\underline{A}^2 + 80\underline{A}^3 - \frac{5}{9}\underline{A}^6 - \frac{1}{3}\underline{A}^8 - \frac{4}{3}\underline{A}^9$$

$$= \frac{2917}{1281}\underline{A}^0 + \frac{1003}{427}(\underline{A}^4 + 9\underline{A}^2 + 28\underline{A}^3) + \frac{409}{1281}(\underline{A}^5 + 16\underline{A}^2 + 48\underline{A}^3)$$

i.e.,

$$\underline{b} = \frac{2917}{1281}\underline{A}^0 + \frac{1600}{1281}\underline{A}^2 + \frac{1404}{1281}\underline{A}^3 + \frac{1003}{427}\underline{A}^4 + \frac{409}{1281}\underline{A}^5$$

$$+ \frac{5}{9}\underline{A}^6 + \frac{1}{3}\underline{A}^8 + \frac{4}{3}\underline{A}^9 \qquad \text{------ (13)}$$

Pricing out the non-basic columns, we find that the vector $\underline{A}^1$ qualifies for entry into the basis.

Now

$$\hat{B} = (P^o,\ P^4,\ P^5)$$

$$= (A^o,\ A^4 + 9A^2 + 28A^3,\ A^5 + 16A^2 + 48A^3)$$

and

$$\bar{P}^1 = \hat{B}^{-1}P^1 = (-155/427,\ 304/427,\ -291/854)^T$$

Therefore,

$$\underline{A}^1 = -155/427\underline{A}^0 + 408/427\underline{A}^2 + 1528/427\underline{A}^3 + 304/427\underline{A}^4 - 291/854\underline{A}^5 \quad -- \quad (14)$$

From (13) and (14) we see that the column $\underline{A}^3$ drops out of the basis. Since $\underline{A}^3 \epsilon S_1$ is key; therefore we first make column $\underline{A}^4$ key replacing $\underline{A}^3$. In this case, the inverse associated with the new key variables in S_1 is

$$V^{(1)^{-1}} = \begin{bmatrix} 2/7 & -1/28 \\ 1/7 & 3/28 \end{bmatrix} .$$

The rearrangement of key variables in S_1 introduces change in the basis for the reduced system which now becomes

$$B_{(1)}^{-1} = \begin{bmatrix} -45/854 & 18/427 & 517/854 \\ 332/427 & 16/427 & -664/427 \\ -59/427 & 9/854 & 118/427 \end{bmatrix}$$

obtained on premultiplication of $\hat{B}^{-1}$ with the matrix

$$M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 28 & 48 \\ 0 & 0 & 1 \end{bmatrix} . \qquad \text{(Note: } \begin{matrix} x_{2/4} = -28, \\ \lambda_{2/5} = -48 \end{matrix} \text{)}$$

We find

$$B_{(1)}^{-1}\ P^1 = (-155/427,\ 1528/427,\ -291/854)^T$$

Therefore, pivoting on the second component, the new inverse basis becomes

$$B^{-1}_{(2)} = \begin{bmatrix} 1 & 155/1528 & 0 \\ 0 & 427/1528 & 0 \\ 0 & 291/3056 & 1 \end{bmatrix} B^{-1}_{(1)}$$

$$= \begin{bmatrix} 5/191 & 23/382 & 171/382 \\ 83/382 & 19/382 & -166/382 \\ -49/764 & 21/764 & 49/382 \end{bmatrix}$$

where now

$$B_{(2)} = (P^0\ ,\ P^1\ ,\ P^5)\quad ;\ P^0 = A^0;\ P^1 = A^1,\ P^5 = A^5 + \frac{4}{7} A^2 - \frac{12}{7} A^4\ .$$

We again compute the pricing vector and find

$$(\pi;\mu^{(1)};\mu^{(2)}) = (5/191,\ 23/382,\ 171/382;\ -63/191,\ 7/1528;\ 449/573,\ 281/573,\ -227/1146)\ .$$

All the columns price out optimally, and hence we have the optimal solution. We find

$$Q_{(2)} = \underline{b} - 5/7\underline{A}^2 - 20/7\underline{A}^4 - 5/9\underline{A}^6 - 1/3\underline{A}^8 - 4/3\underline{A}^9$$

$$= (113/21,\ 256/21,\ 71/21)^T$$

where

$$(x_2,\ x_4) = (5/7,\ 20/7)$$

and

$$(x_6,\ x_8,\ x_9) = (5/9,\ 1/3,\ 4/3)\ .$$

Thus

$$B^{-1}_{(2)} Q_{(2)} = (2737/1146,\ 117/382,\ 971/2292)^T$$

Therefore,

$$\underline{b} - 5/7\underline{A}^2 - 20/7\underline{A}^4 - 5/9\underline{A}^6 - 1/3\underline{A}^8 - 4/3\underline{A}^9$$

$$= \quad 2737/1146\underline{A}^0 + 117/382\underline{A}^1 + 971/2292\ (\underline{A}^5 + 4/7\underline{A}^2 - 12/7\underline{A}^4)$$

i.e.,

$$\underline{b} = 2737/1146\underline{A}^0 + 117/382\underline{A}^1 + 548/573\underline{A}^2 + 407/191\underline{A}^4 + 971/2292\underline{A}^5 + \\ + 5/9\underline{A}^6 + 1/3\underline{A}^8 + 4/3\underline{A}^9 \quad ,$$

from which the optimal solution reads as follows:

$$x_0 = 2737/1146,\ x_1 = 117/382,\ x_2 = 548/573,\ x_4 = 407/191,$$

$$x_5 = 971/2292,\ x_6 = 5/9,\ x_8 = 1/3,\ x_9 = 4/3,\ x_3 = x_7 = x_{10} = 0 \ .$$

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[+] After finishing this work, these two references, having points of similarity as adopted in this paper, were brought to the author's attention by Mr. Paul Rech.

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13 ABSTRACT

The paper [1] by Dantzig and Wolfe suggested the need for developing new techniques for solving linear programming problems with a special matrix structure. A number of techniques have appeared since then. In this report, an algorithm for solving a structured linear programming problem with a very large number of blocks is given. The main feature of the method as described in [3] is to carry out the computation with the help of a smaller basis whose order is equal to the number of linking equations coupling together the various blocks.

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